

# PARTITIONS OF UNITY AND NEW OBSTRUCTIONS FOR GABOR FRAMES

KARLHEINZ GRÖCHENIG

ABSTRACT. We derive new obstructions for Gabor frames. This note explains and proves the computer generated observations of Lemvig and Nielsen [3].

## 1. INTRODUCTION

Given  $\alpha, \beta > 0$  and  $g \in L^2(\mathbb{R})$ , let  $\mathcal{G}(g, \alpha, \beta) = \{e^{2\pi i \beta l \cdot} g(\cdot - \alpha k) : k, l \in \mathbb{Z}\}$  be the Gabor system with window  $g$  and lattice parameters  $\alpha$  and  $\beta$ . The basic question is when  $\mathcal{G}(g, \alpha, \beta)$  generates a frame (called a *Gabor frame*), i.e., when there exist  $A, B > 0$  such that

$$A\|f\|_2^2 \leq \sum_{k,l \in \mathbb{Z}} |\langle f, e^{2\pi i \beta l \cdot} g(\cdot - \alpha k) \rangle|^2 \leq B\|f\|_2^2 \quad \text{for all } f \in L^2(\mathbb{R}).$$

See [2] for a recent survey of Gabor frames with background and a collection of references.

For the characterization of Gabor frames the Zak transform is a standard tool. It is defined as

$$Z_\alpha f(x, \xi) = \sum_{r \in \mathbb{Z}} g(x - \alpha r) e^{2\pi i r \xi}.$$

Given  $p, q \in \mathbb{N}$  with  $p \leq q$  and  $\alpha\beta = \frac{p}{q}$ , we define the matrix  $P(x, \xi)$  with entries

$$(1) \quad P(x, \xi)_{kl} = Z_{\alpha q} g(x + \alpha l + \frac{k}{\beta}, \xi) = Z_{\alpha q}(x + \alpha(l + \frac{qk}{p}), \xi).$$

By (quasi-)periodicity of  $Z$  we may restrict the index set to  $k = 0, \dots, p-1$ , and  $l = 0, \dots, q-1$ . Thus  $P(x, \xi)$  is a  $p \times q$ -matrix.

Lyubarski and Nes [4] gave the following characterization of Gabor frames over rational lattices. We assume that the window is in the Feichtinger algebra  $g \in M^1(\mathbb{R})$ , then the Zak transform and the matrix-valued function  $P$  are continuous.

**Lemma 1.** *Assume that  $\alpha\beta = \frac{p}{q} \in \mathbb{Q}$  with relatively prime  $p, q$ , and  $g \in M^1(\mathbb{R})$ , and let  $P$  be the corresponding family of  $p \times q$ -matrices. Then  $\mathcal{G}(g, \alpha, \beta)$  is a frame, if and only if  $P(x, \xi)$  has rank  $p$  for all  $x, \xi \in \mathbb{R}^2$ .*

The following proposition is a new obstruction for Gabor frames. It explains rigorously some of the observations obtained in [3] with the help of computer algebra.

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**Proposition 2.** Assume that  $g \in M^1(\mathbb{R})$  generates a partition of unity

$$(2) \quad \sum_{s \in \mathbb{Z}} g(x - s) = 1 \quad \text{for all } x \in \mathbb{R}.$$

Let  $m, n, r \in \mathbb{N}$ ,  $j = 1, \dots, r-1$ , such that  $(r-1)m + 1 < rn + j < rm$  and  $rn + j$  and  $rm$  are relatively prime.

If  $\alpha = \frac{1}{m}$  and  $\beta = n + \frac{j}{r}$ , then  $\mathcal{G}(g, \alpha, \beta)$  is not a frame.

*Proof.* In this case  $\alpha\beta = \frac{rn+j}{rm}$ ,  $p = rn + j$ ,  $q = rm$ , and  $\alpha q = r$ . We will show that the matrices  $P(x, 0)$  have rank smaller than  $p = rn + j$  for all  $x$  and thus violate the condition of Lyubarski and Nes. Note that

$$P(x, 0)_{k,l} = Z_r g(x + \alpha l + \frac{k}{\beta}, 0) = \sum_{s \in \mathbb{Z}} g(x + \frac{l}{m} + \frac{rk}{rn+j} - rs)$$

is a periodization of  $g$  with period  $r$ .

Now set  $v_l = \sum_{j=0}^{r-1} \delta_{l+jm} \in \mathbb{C}^q$  for  $l = 0, \dots, m-1$ , where  $\delta_k(k) = 1$  and  $\delta_k(s) = 0$  for  $s \neq k$ . Clearly these vectors are linearly independent. Then

$$\begin{aligned} (P(x, 0)v_l)_k &= \sum_{j=0}^{r-1} P(x, 0)_{k, l+jm} \\ &= \sum_{j=0}^{r-1} \sum_{s \in \mathbb{Z}} g(x + \frac{l+jm}{m} + \frac{k}{\beta} - rs) \\ &= \sum_{s \in \mathbb{Z}} g(x + \frac{l}{m} + \frac{k}{\beta} - s) = 1 \end{aligned}$$

for  $k = 0, \dots, p-1$  and  $l = 0, \dots, m-1$ . For the last equality we have used hypothesis (2).

Setting  $e = (1, \dots, 1)^T \in \mathbb{C}^p$ , we have found  $m$  linearly independent vectors  $v_l$  such that  $P(x, 0)v_l = e$ . Consequently the vectors  $v_0 - v_l, l = 1, \dots, m-1$  are in the kernel of  $P(x, 0)$ . Since they are also linearly independent, we know that  $\dim(\ker P(x, 0)) \geq m-1$ . We obtain that

$$\begin{aligned} \text{rank}(P(x, 0)) &= rm - \dim(\ker P(x, 0)) \\ &\leq rm - (m-1) = (r-1)m + 1 < rn + j = p, \end{aligned}$$

by assumption on  $m, n, r$ . Thus the condition of Lemma 1 is violated and  $\mathcal{G}(g, \alpha, \beta)$  cannot be a frame.  $\square$

**REMARKS:** 1. Lemvig and Nielsen [3] observed that for the linear spline  $B_2 = \chi_{[-1/2, 1/2]} * \chi_{[-1/2, 1/2]}$  the lattices  $\frac{1}{2m+1}\mathbb{Z} \times \frac{n+1}{2}\mathbb{Z}$  do not generate a Gabor frame. With Proposition 2 this is now a rigorous result.

To put the observations of [3] into context, let

$$\mathcal{F}(g) = \{(\alpha, \beta) \in \mathbb{R}_+^2 : \mathcal{G}(g, \alpha, \beta) \text{ is a frame}\}$$

be the *frame set* of  $g$ . This is the set of all rectangular lattices  $\alpha\mathbb{Z} \times \beta\mathbb{Z}$  that generate a Gabor frame with window  $g$ . Proposition 2 says that the points  $(\frac{1}{m}, n + \frac{j}{r})$  near the hyperbola  $\alpha\beta = 1$  do not belong to the frame set  $\mathcal{F}(g)$ . In addition, Lemvig

and Nielsen observed that many of these points are not isolated points in the complement of  $\mathcal{F}(g)$ . Their observations destroy the initial hope that the frame set with respect to  $B$ -spline windows possesses a simple structure. In fact, at this time it seems that the complexity of the frame set of  $B$ -spline windows resembles more that of the characteristic function  $\chi_{[0,1]}$  determined in the stunning work of Dai and Sun [5].

2. The partition-of-unity condition (2) is a well known obstruction in Gabor analysis. It was already observed by Del Prete [1] that  $\mathcal{G}(g, \alpha, \beta)$  fails to be a frame when  $\beta = 2, 3, \dots$  and  $\alpha > 0$  is arbitrary. It comes as a surprise that this condition excludes so many more lattices. In fact, for fixed  $n \in \mathbb{N}, n \geq 2$  and  $m = n + 1$ , Proposition 2 excludes a countable set of points of the form  $(\frac{1}{n+1}, n + \frac{j}{r}), r \in \mathbb{N}$  with accumulation point  $(\frac{1}{n+1}, n + 1)$  from  $\mathcal{F}(g)$ .

## REFERENCES

- [1] V. Del Prete. Estimates, decay properties, and computation of the dual function for Gabor frames. *J. Fourier Anal. Appl.*, 5(6):545–562, 1999.
- [2] K. Gröchenig. The mystery of Gabor frames. *J. Fourier Anal. Appl.*, 20(4):865–895, 2014.
- [3] J. Lemvig and K. H. Nielsen. A counterexample to the B-spline conjecture for Gabor frames. arXiv:1507.03982.
- [4] Y. Lyubarskii and P. G. Nes. Gabor frames with rational density. *Appl. Comput. Harmon. Anal.*, 34(3):488–494, 2013.
- [5] X.-R. Dai and Q. Sun. The *abc*-problem for Gabor systems. To appear in *Mem. Amer. Math. Soc.*, arxiv.org/pdf/1304.7750.pdf.

FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, NORDBERGSTRASSE 15, A-1090 VIENNA, AUSTRIA

*E-mail address:* karlheinz.groechenig@univie.ac.at